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SINGULAR POINT TRANSFORMATIONS IN TWO COMPLEX VARIABLES.*

BY GUY ROGER CLEMENTS.

This paper will be devoted to a detailed study of the transformation $T: x = f(u, v), y = \varphi(u, v)$,

(a) $f(u, v)$ and $\varphi(u, v)$ denoting functions of the complex variables u and v , singlevalued and analytic throughout a neighborhood R of $u = 0, v = 0$;

(b) $f(0, 0) = 0, \quad \varphi(0, 0) = 0$;

(c) $J(u, v) \equiv f_u \varphi_v - f_v \varphi_u \neq 0, \quad J(0, 0) = 0$.

To any point (u, v) of R there corresponds, under T , one and but one point (x, y) of the xy -space. The totality of points (x, y) which thus correspond to points (u, v) of R constitute a finite region $\bar{R}$ of the xy -space. In general more than one point (u, v) will yield the same pair of values for x and y . We shall count a point (x, y) but once, thus regarding it as completely characterized by its coördinates, and shall seek an inverse transformation

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y),$$

which will put in evidence all those points (u, v) of R which correspond under T to any point of $\bar{R}$ with coördinates (x, y) . Thus the functions $\bar{f}(x, y)$ and $\bar{\varphi}(x, y)$ are to be defined only in the points of $\bar{R}$ and there as singlevalued or as multiple-valued functions.

We shall distinguish two types of neighborhood.

If $x_1, x_2, \dots, x_n$ denote n independent complex variables, any region which constitutes at least the totality of all points $(x_1, x_2, \dots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \quad |x_2 - a_2| < \epsilon, \dots, \quad |x_n - a_n| < \epsilon,$$

for some $\epsilon > 0$, is called a *complete* neighborhood of the point $(a_1, a_2, \dots, a_n)$.

Any region which satisfies the conditions that

(1) for every $\epsilon > 0$, it contains points $(x_1, x_2, \dots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \quad |x_2 - a_2| < \epsilon, \quad \dots, \quad |x_n - a_n| < \epsilon,$$

(2) for no $\epsilon > 0$ is every point $(x_1, x_2, \dots, x_n)$ for which

$$|x_1 - a_1| < \epsilon, \quad |x_2 - a_2| < \epsilon, \quad \dots, \quad |x_n - a_n| < \epsilon,$$

a point of the region,

is called a *partial* neighborhood of the point $(a_1, a_2, \dots, a_n)$.

* The results of this paper were in part announced in the Bulletin of the American Mathematical Society, 2d Series, Vol. XVIII (1912), pp. 454-456.

For purposes of simplicity the neighborhood of the point $(0, 0, \dots, 0)$ is considered throughout this paper. In what follows the unqualified word *neighborhood* is used to mean *complete neighborhood*. R of the transformation T is a complete neighborhood.

2. Factoring.

A transformation can often be factored to advantage, that is, be replaced by the succession of two or more transformations each of which is simpler or better understood than the given one. We illustrate with a very useful example:

$$t: \quad x = h^m, \quad y = h^n(g^p + c),$$

where $h = h(u, v)$, $g = g(u, v)$ are single-valued and analytic functions of u and v in the point $(0, 0)$ and vanish there, where m, n, p are positive integers and where c is a constant. t is equivalent to the $n + 4$ transformations:

$$\begin{array}{ll} u_0 = h(u, v), & v_0 = g(u, v), \\ u_1 = u_0, & v_1 = v_0^p, \\ u_2 = u_1, & v_2 = u_1 v_1, \\ \cdot & \cdot \\ u_{n+1} = u_n, & v_{n+1} = u_n v_n, \\ u_{n+2} = u_{n+1}, & v_{n+2} = v_{n+1} + c u_{n+1}^n, \\ x = u_{n+2}^m, & y = v_{n+2}. \end{array}$$

This makes use of the two types

$$r: \quad x = u, \quad y = uv; \quad \text{and} \quad s: \quad x = u, \quad y = v^n,$$

which we shall discuss in detail. We note that each is wholly independent of the coefficients of t . The only other factor which is not surely one-to-one and analytic both ways is

$$(1) \quad u_0 = h(u, v), \quad v_0 = g(u, v).$$

We can retrace our steps, build one inverse on another and thus get a picture of u_0 and v_0 as functions of x and y , and of the relations between their regions of definition. If the inverse of (1) can be found, the entire inverse can be built up and the solution is complete.

We examine the auxiliary transformations r and s .

$$r: \quad x = u, \quad y = uv; \quad J = u.$$

For definiteness in the geometric image consider first the special case

that u and v have only real values. Let R be the region bounded by the lines $u = \pm 1$, $v = \pm 1$. The corresponding region $\bar{R}$ in the xy -plane is bounded by the lines $x = \pm y$, $x = \pm 1$; any line $v = au$ goes over into the parabola $y = ax^2$. To the locus $u = 0$ corresponds the point $x = 0$, $y = 0$ and conversely. That is, the inverse of r explodes the point $x = 0$, $y = 0$ into the line segment $u = 0$ of R . There is no point in $\bar{R}$ for which $x = 0$, $y \neq 0$. To every point of $\bar{R}$ other than $(0, 0)$ there corresponds one and but one point (u, v) of R determined by the relations $u = x$, $v = y/x$. The right members of these equations are single-valued and analytic at every point of $\bar{R}$ except at the origin where we have seen geometrically what is happening.

For the less simple transformation

$$r': \quad x = u, \quad y = u^n v, \quad J = u^n,$$

the discussion is the same except that $\bar{R}$ is bounded by the curves $y = \pm x^n$, $x = \pm 1$. Further, r' is equivalent to r repeated n times.

To see the analogous results when u and v are allowed complex values let the definition of R be $|u| < 1$, $|v| < 1$. Then since $|x| = |u|$, $|y| = |u| \cdot |v|$, it is evident that the circles $|u| = 1$, $|v| = 1$ go over into the circles $|x| = 1$, $|y| = 1$, and that if a point in the x -plane be marked inside this circle $|x| = 1$, then a corresponding point y must be chosen for which y is less in absolute value than this x if the resultant point (x, y) is to be a point of $\bar{R}$. This shows that $\bar{R}$ is only a partial neighborhood of the origin. It follows as for the case of real variables above that the inverse is well defined, single-valued and analytic in every point of $\bar{R}$ except the origin and that the origin has an infinitely many-valued inverse $u = 0$, $v = v$.

$$s: \quad x = u, \quad y = v^n, \quad J = nv^{n-1}.$$

Let R be defined by $|u| < u_0 > 0$, $|v| < v_0 > 0$. Take any point (u', v') of R and let (x', y') denote its image in $\bar{R}$. There are n points (u, v) of R which have the same image (x', y') , namely those for which $u = u'$, v equals the product of v' by any one of the n th roots of unity. Hence conversely to a point of $\bar{R}$ with coördinates (x', y') there correspond n distinct points (u, v) of R , the relationship being defined by the equations

$$u = x, \quad v = y^{1/n}.$$

$y^{1/n}$ is an n -valued continuous function of x and y whose branches come together in $x = x$, $y = 0$.

Since we shall use power series freely, the definition $|u| < u_0$, $|v| < v_0$ for R is the one of practical use for this paper. For the problem immediately in hand however, we might have taken the u -space arbitrarily and for the

by $u = h(u_1)$, $v = v_1$, whence

$$f[h(u_1), v_1] = u_1^m + A_1(u_1)v_1 + A_2(u_1)v_1^2 + \dots$$

We assume that $f(u, v)$ has this normalized form and consequently have before us

$$(2) \quad f(u, v) = u^m + A_1(u)v + A_2(u)v^2 + \dots,$$

$$(3) \quad \varphi(u, v) = B_0(u) + B_1(u)v + B_2(u)v^2 + \dots,$$

where

$$(4) \quad B_0(u) = b_p u^p + b_{p+1} u^{p+1} + \dots$$

Perform as many times as possible the process of factoring powers of $f(u, v)$ out of $\varphi(u, v)$ as indicated in the identity:

$$(5) \quad \begin{aligned} \varphi(u, v) &= f^{n_1} \{C_1 + f^{n_2} [C_2 + \dots + f^{n_k} (C_k + g_k(u, v)) \dots]\} \\ &= C_1 f^{n_1} + C_2 f^{n_1+n_2} + \dots + C_k f^{n_1+\dots+n_k} + f^{n_1+\dots+n_k} g_k(u, v) \end{aligned}$$

$$(6) \quad = F(f) + f^N g_k(u, v),$$

where $C_1, C_2, \dots, C_k$ are constants which, with the possible exception of C_k , are different from zero, where $n_1, n_2, \dots, n_k$ are positive integers whose sum is N , where $F(f)$ is a polynomial in f of degree not greater than N , and where $g_k(0, 0) = 0$, but $f(u, v)$ is not a factor of $g_k(u, v)$ in the point $(0, 0)$.

To compute the coefficients C_i of F , assume for $g_k(u, v)$ a formal expansion $d_0(u) + d_1(u)v + d_2(u)v^2 + \dots$, and combine this with (2) and (5) to derive

$$(7) \quad \begin{aligned} \varphi(u, v) &= C_1 [u^m + A_1(u)v + \dots]^{n_1} + \dots + C_k [u^m + A_1(u)v + \dots]^{n_1+\dots+n_k} \\ &\quad + [u^m + A_1(u)v + \dots]^{n_1+\dots+n_k} [d_0(u) + d_1(u)v + \dots] \\ &= C_1 u^{mn_1} + C_2 u^{m(n_1+n_2)} + \dots + C_k u^{m(n_1+\dots+n_k)} + \text{higher powers of} \\ &\quad u + \text{terms involving } v \text{ also.} \end{aligned}$$

Compare (3) with (7) and equate the coefficients of like powers of u in the terms involving u alone. This gives

$$(8) \quad C_i = b_{m(n_1+n_2+\dots+n_i)}, \quad i = 1, 2, \dots, k.$$

But $\sum_{j=1}^{\infty} b_j u^j$ is absolutely and uniformly convergent, and consequently, for any N , $F(f(u, v))$ is single-valued and analytic in a suitable neighborhood of the origin. It follows next that the difference of two analytic functions,

$$f^N g_k(u, v) = \varphi(u, v) - F(f),$$

is likewise analytic in this same region. Consequently we may write the transformation T :

$$x = f(u, v), \quad y = \varphi(u, v) \equiv F(f) + f^N g_k(u, v),$$

whence we have formally

$$J(u, v) \equiv f_u \varphi_v - f_v \varphi_u \equiv f^N (f_u g_{kv} - f_v g_{ku}).$$

This, together with the hypothesis that $J(u, v) \not\equiv 0$, establishes two things: (1) N is not greater than the greatest power of $f(u, v)$ that is contained as a factor in $J(u, v)$, therefore N is finite; (2) $J(f, g_k) \not\equiv 0$ and hence, in particular, $g_k(u, v)$ is not a constant. In the identity (5), $1 \leq k \leq N$, hence $g_k(u, v)$ is the result of finite processes of the sort there indicated, and is therefore single-valued and analytic at the origin.

The following factoring for T is now apparent:

$$\begin{aligned} u_0 &= f(u, v), & v_0 &= g_k(u, v), \\ u_1 &= u_0, & v_1 &= u_0^{n_k} v_0, \\ u_2 &= u_1, & v_2 &= v_1 + C_k u_1^{n_k}, \\ &\cdot & &\cdot \\ u_{2i+1} &= u_{2i}, & v_{2i+1} &= (u_{2i})^{n_k} v_{2i}, \\ u_{2i+2} &= u_{2i+1}, & v_{2i+2} &= v_{2i+1} + C_{k-1} (u_{2i+1})^{n_{k-1}}, \\ &\cdot & &\cdot \\ u_{2k-1} &= u_{2k-2}, & v_{2k-1} &= (u_{2k-2})^{n_1} v_{2k-2}, \\ x &= u_{2k-1}, & y &= v_{2k-1} + C_1 (u_{2k-1})^{n_1}, \end{aligned}$$

where, as we have seen in section 2, any transformation

$$u_{2i+1} = u_{2i}, \quad v_{2i+1} = (u_{2i})^{n_k} v_{2i}$$

is equivalent to n_{k-i} transformations of the type $x = u, y = uv$.

If $g_k(u, v)$ is not a factor of $f(u, v)$ in the point $(0, 0)$, the factoring of T is *complete*. If $g_k(u, v)$ is a factor of $f(u, v)$ in this point we must further factor

$$a: \quad u_0 = f(u, v), \quad v_0 = g_k(u, v).$$

The Jacobian $J(f, g_k)$ can not be identically zero in R , hence all the previous discussion for Case B is valid for the transformation a . Corresponding to a in the factoring of T we shall have here a transformation

$$a': \quad v_0' = g_k(u, v), \quad u_0' = f_p(u, v),$$

where $g_k(u, v)$ is not a factor of $f_p(u, v)$ in the point $(0, 0)$. We must finally, in a finite number of steps, come to a transformation

$$a'': \quad U = F(u, v), \quad V = G(u, v),$$

for which $F(u, v)$ is not a factor of $G(u, v)$ and $G(u, v)$ is not a factor of $F(u, v)$ in the point $(0, 0)$. For suppose the Weierstrassian preparation theorem* gives

$$f(u, v) \equiv [u^r + c_1(v)u^{r-1} + \cdots + c_r(v)]C(u, v),$$

$$g_k(u, v) \equiv [u^s + d_1(v)u^{s-1} + \cdots + d_s(v)]D(u, v),$$

where neither $C(u, v)$ nor $D(u, v)$ is zero in R . If $g_k(u, v)$ is a factor of $f(u, v)$ while $f(u, v)$ is not a factor of $g_k(u, v)$, it follows that s is less than r . Similarly, if $f_p(u, v)$ is a factor of $g_k(u, v)$ and if

$$f_p(u, v) \equiv [u^t + e_1(v)u^{t-1} + \cdots + e_t(v)]E(u, v),$$

then t must be less than s which is less than r . Proceeding in this way, we see that we must in not more than r steps arrive at a transformation for which one of the Weierstrassian polynomials is of degree unity, unless a transformation satisfying the conditions of a'' is earlier obtained. For definiteness, suppose we have

$$u'' = \bar{f}(u, v), \quad v'' = \bar{g}(u, v),$$

where

$$\bar{f}(u, v) \equiv [u + A(v)]H(u, v), \quad H(0, 0) \neq 0,$$

and where $\bar{f}(u, v)$ is a factor of $\bar{g}(u, v)$ in the point $(0, 0)$. Applying the initial discussion for Case *B* (which is again valid since the Jacobian for this transformation can not vanish identically) we secure a transformation

$$u_0'' = \bar{f}(u, v), \quad v_0'' = \bar{g}_j(u, v),$$

where $\bar{f}(u, v)$ is not a factor of $\bar{g}_j(u, v)$ in the point $(0, 0)$. If $\bar{g}_j(u, v)$ is a factor of $\bar{f}(u, v)$ in this point, it must also be linear in u in the neighborhood of the origin, and $\bar{f}(u, v)$ and $\bar{g}_j(u, v)$ must be equivalent. This is in direct contradiction to the fact that $\bar{f}(u, v)$ is not a factor of $\bar{g}_j(u, v)$ in the point $(0, 0)$. This establishes the existence of a transformation a'' . We have therefore proved

THEOREM 1. *The transformation T can always be replaced by a transformation*

$$T': \quad U = F(u, v), \quad V = G(u, v),$$

where $F(u, v)$ and $G(u, v)$ are single-valued and analytic in the point $(0, 0)$ and both vanish there, but where neither is a factor of the other in this point, followed by a finite number m of transformations of the form

$$x = u, \quad y = uv$$

* Weierstrass, *Abhandlungen aus der Funktionenlehre*, p. 105; Goursat, *Cours d'Analyse*, vol. 2, Sec. 356; Bliss, *Bulletin of the American Mathematical Society*, vol. 16 (1910), p. 356; Macmillan, *ibid.*, vol. 17 (1910), p. 116.

combined with not more than m transformations of the form

$$x = u, \quad y = v + cu^n.$$

The transformation T' either is one-to-one and analytic both ways or is again of type T .

4. Case A.

Because of the preceding section, the solution of our problem will be attained when we have a discussion of the nature of the inverse for Case A. In discussing Case A, we note that any transformation T is of one of the following two classes:

(I) At least one element of the determinant J , say f_u , is not zero at the origin.

(II) Every element of the determinant J is zero at the origin.

In considering Case IA we therefore have:

$$T''': \quad x = f(u, v), \quad y = \varphi(u, v),$$

where

$$f_u(0, 0) \neq 0$$

and where $f(u, v)$ is not a factor of $\varphi(u, v)$ in the point $(0, 0)$.

This is the special case $n = 2$ of the following theorem.* The transformation

$$x_1 = f_1(y_1, \dots, y_n), \quad x_2 = f_2(y_1, \dots, y_n), \quad \dots, \quad x_n = f_n(y_1, \dots, y_n),$$

where

(1) f_i is a single-valued and analytic function of $y_1, \dots, y_n$ in the point $(0, \dots, 0)$ and vanishes there ($i = 1, 2, \dots, n$);

$$J_1(y_1, \dots, y_n) \equiv \frac{D(f_1, \dots, f_n)}{D(y_1, \dots, y_n)} = 0 \text{ when } (y) = 0,$$

.

(2) $J_{m-1}(y_1, \dots, y_n) \equiv \frac{D(J_{m-2}, f_2, \dots, f_n)}{D(y_1, \dots, y_n)} = 0 \text{ when } (y) = 0,$

$$J_m(y_1, \dots, y_n) \equiv \frac{D(J_{m-1}, f_2, \dots, f_n)}{D(y_1, \dots, y_n)} \neq 0 \text{ when } (y) = 0;$$

has an m -valued continuous inverse, defined throughout a complete neighborhood of $(x) = 0$. This inverse is analytic, with m distinct branches except along a complex $(n - 1)$ -dimensional locus, where it is continuous and less than m -valued. Finally, $(y) = 0$ when $(x) = 0$.

* G. R. Clements, Bulletin of the American Mathematical Society, loc. cit., p. 454; Transactions, ibid., vol. XIV (1913), p. 341.

T'' evidently fulfills condition (1) of this theorem; to see that it also satisfies condition (2), make the factoring

$$a: \quad u_0 = f(u, v), \quad v_0 = v,$$

$$b: \quad x = u_0, \quad y = \Phi(u_0, v_0).$$

The factor a is one-to-one and analytic both ways, with inverse

$$u = h(u_0, v_0), \quad v = v_0.$$

Since $f_u(0, 0) \neq 0$, $f(u, v)$ is irreducible in R . $\Phi(0, v_0) \neq 0$, for if $\varphi[h(u_0, v_0), v_0]$ is zero wherever $u_0 = f(u, v)$ (irreducible) is zero, then $\varphi(u, v)$ admits $f(u, v)$ as a factor contrary to hypothesis. Therefore $\Phi(0, v_0) \neq 0$ and there is a first coefficient b_m in its series expansion,

$$\Phi(u_0, v_0) = \sum_{n=0}^{\infty} b_n(u_0)v_0^n,$$

which is not zero when $u_0 = 0$. The vanishing of J_i in the point $(0, 0)$ is invariant under the transformation a , therefore

$$J_1(0, 0) = \cdots = J_{m-1}(0, 0) = 0, \quad J_m(0, 0) \neq 0.$$

Hence for this transformation IA there exists an m and a sequence of functions satisfying condition (2) of the theorem quoted above, and all the requirements of that theorem are fulfilled. Since the inverse can not be single-valued and analytic,* m must be an integer not less than two. This proves

THEOREM 2. *If the transformation T satisfies the further conditions*

$$(1) \quad f_u(0, 0) \neq 0,$$

$$(2) \quad f(u, v) \text{ not a factor of } \varphi(u, v) \text{ in the point } (0, 0),$$

then, for the sequence of functions

$$J_1(u, v) = \frac{D(f, \varphi)}{D(u, v)}, \quad \cdots, \quad J_{n+1}(u, v) = \frac{D(f, J_n)}{D(u, v)}$$

there exists an integer $m \geq 2$ such that

$$J_1(0, 0) = \cdots = J_{m-1}(0, 0) = 0, \quad J_m(0, 0) \neq 0,$$

and for the transformation there exists an m -valued continuous inverse defined throughout a complete neighborhood of $x = 0, y = 0$. This inverse is analytic, with m distinct determinations except along a complex one-dimensional locus, where it is continuous and less than m -valued. Finally $u = 0, v = 0$ when $x = 0, y = 0$.

* Bulletin, loc. cit., p. 452, Theorem 1.

Further factoring for a special class of transformations IA is exhibited in the following theorem.*

THEOREM 3. *If in the transformation of Theorem 2, $J_{m-1}(u, v)$ is a factor in the point $(0, 0)$ of every $J_n(u, v)$ with smaller subscript, then T can be replaced by transformations one-to-one and analytic both ways, combined with one transformation of the form $x = u, y = v^m$.*

Replace T by the factors

$$\begin{aligned} a: & \quad u_0 = f(u, v), & v_0 &= v, \\ b: & \quad x = u_0, & y &= \Phi(u_0, v_0). \\ \text{Set} & \\ c: & \quad u_0 = u_1, & v_0 &= a_0(u_1) + a_1(u_1)v_1, \end{aligned}$$

where a_0 and a_1 are functions to be determined. If

$$\Phi(u_0, v_0) = \sum_{n=0}^{\infty} b_n(u_0)v_0^n,$$

formal substitution yields

$$\begin{aligned} \Phi[u_1, a_0(u_1) + a_1(u_1)v_1] &= \sum_{n=0}^{\infty} b_n \cdot (a_0 + a_1v_1)^n = \sum_{n=0}^{\infty} b_n a_0^n \\ (9) \quad &+ a_1v_1 \sum_{n=0}^{\infty} n b_n a_0^{n-1} + (a_1^2/2)v_1^2 \sum_{n=0}^{\infty} n(n-1)a_0^{n-2} + \dots = \Phi(u_1, a_0) \\ &+ a_1 \Phi_{a_0}(u_1, a_0)v_1 + \dots + (a_1^n/n!) \Phi_{a_0^n}(u_1, a_0)v_1^n + \dots \end{aligned}$$

$\Phi_{a_0^{m-1}}(u_1, a_0)$ is single-valued and analytic in the point $u_1 = 0, a_0 = 0$ and vanishes there; and further has a derivative with respect to a_0 which is different from zero in this point. Consequently there exists a function $a_0(u_1)$, single-valued and analytic in the point $u_1 = 0$ and vanishing there, for which

$$\Phi_{a_0^{m-1}}[u_1, a_0(u_1)] \equiv 0.$$

Because of the hypothesis on the J_n , $\Phi_{a_0^{m-1}}$ is a factor of every partial derivative of Φ with respect to a_0 of order lower than $m-1$. Consequently this determination of $a_0(u_1)$ requires that the coefficients of $v_1, \dots, v_1^{m-2}$ be also identically zero. If, therefore, we assign to $a_1(u_1)$ an arbitrary analytic function of u_1 which does not vanish in R , the transformation c is one-to-one and analytic both ways, transforms origin into origin and transforms $\Phi(u_0, v_0)$ into an analytic function

$$(10) \quad P(u_1, v_1) \equiv B_0(u_1) + B_m(u_1)v_1^m + B_{m+1}(u_1)v_1^{m+1} + \dots,$$

where

$$B_m(u_1) \equiv \frac{a_1(u_1)^m}{m!} \Phi_{a_0^m}(u_1, a_0(u_1)),$$

* Presented to the American Mathematical Society on October 26, 1912.

is not zero in R . That is, b becomes

$$x = u_1, \quad y = B_0(u_1) + B_m(u_1)v_1^m + \cdots, \quad B_m(0) \neq 0.$$

Make the transformation

$$\begin{aligned} d: \quad & u_2 = u_1, \quad v_2 = v_1 \sqrt[m]{B_m(u_1) + B_{m+1}(u_1)v_1 + \cdots}; \\ s: \quad & u_3 = u_2, \quad v_3 = v_2^m; \\ e: \quad & x = u_3, \quad y = B_0(u_3) + v_3. \end{aligned}$$

Then, in symbols,

$$T = a \cdot c^{-1} \cdot d \cdot s \cdot e, \quad T^{-1} = e^{-1} \cdot s^{-1} \cdot d^{-1} \cdot c \cdot a^{-1}.$$

s is the only one of these transformations which is not one-to-one and analytic both ways. Hence the theorem is proved.

If $m = 2$, the hypotheses of this theorem are satisfied identically.*

We turn next to Case IIA, a part of which is taken care of by the following theorem:† If in the transformation T , $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$, and if R be suitably restricted, then there exists an inverse, defined throughout the (complete) neighborhood of $x = 0$, $y = 0$, everywhere continuous in that neighborhood, finitely multiple-valued but not single-valued, analytic except along a complex one-dimensional locus, and having the value $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

There remain those transformations of Case IIA for which $f(u, v)$ and $\varphi(u, v)$ have a common factor in the point $(0, 0)$. The origin is evidently an explosive point for the inverse; examples indicate that in any other point of $\bar{R}$ there exists a finitely multiple-valued inverse, the order of this multiplicity being in general different in different sub-regions of $\bar{R}$. I hope to give this theory more fully at a later time.

5. Special Transformations.

In section 2 we exhibited at least a partial factoring for any transformation in which either $f(u, v)$ or $\varphi(u, v)$ is the integral power (greater than one) of an analytic function of u and v . The factoring there made

* For this special case $m = 2$, the corresponding theorem for transformations in two and in three real variables has been given respectively by L. S. Dederick, Harvard doctoral thesis (1909), "Certain singularities of transformations of two real variables," p. 124; and by S. E. Urner, "Certain singularities of point transformations in space of three dimensions," Transactions of the American Mathematical Society, Vol. XIII (1912), p. 257, Ex. a. The extension of Mr. Urner's example b of which he says: "There seems to be no theorem in this case analogous to that stated in (a)," is found in my theorem 3. This case $m = 2$ was also discussed for a transformation in n real variables, by Mr. Dederick in the Transactions, Vol. XIV, p. 143.

† G. R. Clements, Transactions, loc. cit., p. 328.

is complete for t if it is complete for

$$u_0 = h(u, v), \quad v_0 = g(u, v).$$

In sections 3 and 4 we have shown that every transformation of Class I is completely factorable. An immediate corollary of these two facts is

THEOREM 4. *A transformation T which has the form*

$$\begin{aligned} x &= f(u) \equiv a_n u^n + a_{n+1} u^{n+1} + \dots \quad (a_n \neq 0), \\ y &= \varphi(u, v), \end{aligned}$$

can be replaced by two transformations, $T = a \cdot b$, where

$$\begin{aligned} a: \quad u_1 &= u \sqrt[n]{a_n + a_{n+1}u + \dots}, & v_1 &= \varphi(u, v), \\ b: \quad x &= u_1^n, & y &= v_1. \end{aligned}$$

(a) has an inverse defined by analytic functions, or is completely factorable.

To illustrate the way in which the inverse can be built up for a transformation of type B which has been completely factored, we derive

THEOREM 5. *If the transformation T has the form*

$$x = f(u, v), \quad y = [f(u, v)]^n g(u, v),$$

where

$$f_u(0, 0) \neq 0, \quad g(0, 0) = 0$$

and where $f(u, v)$ is not a factor of $g(u, v)$ in the point $(0, 0)$, then to the point $x = 0, y = 0$, corresponds the locus $f(u, v) = 0$, and to any other point (x, y)

(1) if $f_u g_v - f_v g_u \neq 0$ when $u = 0, v = 0$, there corresponds one and only one point (u, v) , u and v being defined by functions of x and y , single-valued and analytic in this point;

(2) if $f_u g_v - f_v g_u = 0$ when $u = 0, v = 0$, there correspond in general m points, $m \geq 2$. The inverse is m -valued and analytic in this point unless it be on a certain complex one-dimensional locus, where it is continuous and less than m -valued.

$\bar{R}$ is a partial neighborhood of the point $x = 0, y = 0$.

A factoring is

$$\begin{aligned} a: \quad u_0 &= f(u, v), & v_0 &= g(u, v), \\ b: \quad x &= u_0, & y &= u_0^n v_0. \end{aligned}$$

In case (1) there exists for a an inverse

$$u = \bar{f}(u_0, v_0), \quad v = \bar{\varphi}(u_0, v_0),$$

defined by functions single-valued and analytic in the neighborhood of $u_0 = 0, v_0 = 0$. For b there exists the inverse $u_0 = x, v_0 = y/x^n$, single-valued and analytic in x and y in every point of $\bar{R}$ except the origin (see

section 2). Hence u and v are defined as single-valued and analytic functions of x and y in any point of $\bar{R}$ other than $x = 0, y = 0$. This point explodes into the locus

$$u_0 = f(u, v) = 0.$$

In case (2), a satisfies the hypotheses of Theorem 2. Therefore a has a finitely multiple-valued (suppose m -valued) inverse defined throughout the complete neighborhood of $u_0 = 0, v_0 = 0$, analytic with m distinct determinations except along a complex one-dimensional locus $D(u_0, v_0) = 0$, where it is continuous and less than m -valued. Further u_0 and v_0 are single-valued and analytic in x and y at every point (x, y) of $\bar{R}$ except the origin. Consequently the inverse defines u and v as in general m -valued and analytic in x and y in a point of $\bar{R}$; in points of a complex one-dimensional locus, excluding the origin, the inverse ceases to be analytic but is continuous and less than m -valued; the point $x = 0, y = 0$, explodes into the locus $f(u, v) = 0$.

Because of the nature of b , $\bar{R}$ is in either case the partial neighborhood of $x = 0, y = 0$.

We give the detailed discussion for a special case of type IIA.

THEOREM 6. *If the transformation T has the form*

$$x = f(u, v) \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + \dots,$$

$$y = \varphi(u, v) \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + \dots,$$

where the terms quadratic in u and v are not identically zero for either $f(u, v)$ or $\varphi(u, v)$ and where these quadratic terms have no common factor in the point $(0, 0)$, then there exists a four-valued continuous inverse, defined throughout the complete neighborhood of $x = 0, y = 0$. This inverse is analytic, with four distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than four-valued. Finally, $u = 0, v = 0$ when $x = 0, y = 0$.

Because of the hypotheses on the quadratic forms

$$f_2 \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2$$

and

$$\varphi_2 \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2,$$

it is always possible to secure from T by a linear transformation in x and y whose Jacobian is not zero, a second transformation T'' for which one of the quadratic forms involved is a perfect square while the second quadratic form is not a perfect square and does not admit the square root of the first form as a factor. Then by a linear transformation in u and v whose Jacobian

is not zero, this can be brought to the form

$$\begin{aligned} T': \quad x &= f(u', v') \equiv v'^2 + c_{30}u'^3 + c_{21}u'^2v' + \dots, \\ y &= \varphi(u', v') \equiv d_{20}u'^2 + d_{11}u'v' + d_{02}v'^2 + d_{30}u'^3 + \dots, \end{aligned}$$

where $d_{20} \neq 0$, $d_{11} \neq 0$, $d_{02} \neq 0$. Of course f , φ , the c 's and d 's have a different definition in T' than they have in T of the theorem. The Jacobian for T' is

$$(11) \quad J = -4d_{20}u'v' - 2d_{11}v'^2 + \dots.$$

Seek for the transformation

$$(12) \quad u' = u + sv, \quad v' = r + v,$$

so to determine r and s that the coefficients of the first power of v in the series into which (12) carries the right members of T' , shall be identically zero. If we write T' as

$$\begin{aligned} (13) \quad x &= f(u', v') \equiv a(u') + a'(u')v' + a''(u')v'^2 + \dots, \\ y &= \varphi(u', v') \equiv b(u') + b'(u')v' + b''(u')v'^2 + \dots, \end{aligned}$$

formal substitution of (12) in $f(u', v')$ yields

$$x = f(u', v') = f(u + sv, r + v) = \sum_{n=0}^{\infty} a^{(n)}(u + sv)(r + v)^n,$$

whence the coefficient of v is

$$\begin{aligned} &a_u(u)s + [a_u'(u)sr + a'(u)] + [a_u''(u)sr^2 + a''(u)2r] \\ &\quad + [a_u'''(u)sr^3 + a'''(u)3r^2] + \dots \\ (14) \quad &= s(a_u + a_u'r + a_u''r^2 + a_u'''r^3 + \dots) + (a' + 2a''r + 3a'''r^2 + \dots) \\ &= sf_u(u, r) + f_r(u, r). \end{aligned}$$

Similarly, for φ the coefficient of v is

$$(15) \quad s\varphi_u(u, r) + \varphi_r(u, r).$$

Setting (14) and (15) each equal to zero and eliminating s , we have as a necessary condition for the determination of r

$$(16) \quad f_u(u, r) \cdot \varphi_r(u, r) - f_r(u, r) \cdot \varphi_u(u, r) \equiv J(u, r) = 0.$$

But

$$J(u, r) = -4d_{20}ur - 2d_{11}r^2 + \dots.$$

The solutions r of $J(u, r) = 0$, which are zero when $u = 0$, are had from*

$$(17) \quad \frac{J(u, r)}{-2d_{11}} \equiv [r^2 + A(u)r + B(u)]e^{w(u, r)}, \quad w(0, 0) = 0.$$

* Weierstrass's preparation theorem.

Comparison of the two members of this identity shows that

$$A(0) = 0, \quad A_u(0) = 2d_{20}/d_{11} \neq 0, \quad B(0) = 0, \quad B_u(0) = 0, \quad B_{u^2}(0) = 0.$$

That is, $A(u)$ and $B(u)$ have the form

$$A(u) \equiv (2d_{20}/d_{11})u + hu^2 + h'u^3 + \dots,$$

$$B(u) \equiv mu^3 + m'u^4 + \dots.$$

Solving

$$r^2 + A(u)r + B(u) = 0$$

for r , we obtain

$$(18) \quad r = \frac{1}{2}[-A(u) \pm \sqrt{A^2(u) - 4B(u)}] \\ = \frac{1}{2}[-(2d_{20}/d_{11})u - hu^2 - \dots \pm (2d_{20}/d_{11})u \sqrt{1 + nu + n'u^2 + \dots}].$$

If we take the lower sign with the radical

$$r' = -(2d_{20}/d_{11})u + pu^2 + p'u^3 + \dots;$$

if we take the upper sign

$$(19) \quad r'' = qu^2 + q'u^3 + \dots.$$

To determine s consider in more detail the equation

$$s\varphi_u(u, r) + \varphi_r(u, r) = 0,$$

where r is replaced by $r''(u)$.*

$$\varphi_u(u, r) \equiv b_u + b_u'r + b_u''r^2 + \dots = u(2d_{20} + g'u + g''u^2 + \dots),$$

$$\varphi_r(u, r) \equiv b' + 2b''r + 3b'''r^2 + \dots = u(d_{11} + h'u + h''u^2 + \dots).$$

Therefore

$$s\varphi_u(u, r) + \varphi_r(u, r) = u[s(2d_{20} + g'u + g''u^2 + \dots) + (d_{11} + h'u + \dots)].$$

Since $u \neq 0$, this will be identically zero throughout R if and only if

$$(20) \quad s = \frac{-(d_{11} + h'u + h''u^2 + \dots)}{2d_{20} + g'u + g''u^2 + \dots} = -\frac{d_{11}}{2d_{20}} + k'u + k''u^2 + \dots.$$

This pair of values for r and s also makes $s f_u(u, r) + f_r(u, r)$ identically zero. For

$$J(u, r''(u)) \equiv 0,$$

that is, when r is replaced by $r''(u)$,

$$f_u \varphi_r - f_r \varphi_u \equiv 0.$$

* The use of $r''(u)$ rather than $r'(u)$ is not accidental. If $r'(u)$ be used, s is not defined when $u = 0$.

Since each derivative in the left member of this identity has u as a factor, let

$$\begin{aligned} f_u(u, r''(u)) &\equiv u\bar{f}_u, & f_r(u, r''(u)) &\equiv u\bar{f}_r, \\ \varphi_u(u, r''(u)) &\equiv u\bar{\varphi}_u, & \varphi_r(u, r''(u)) &\equiv u\bar{\varphi}_r, \end{aligned}$$

where

$$\bar{\varphi}_u(0) = 2d_{20} \neq 0.$$

Then

$$J(u, r''(u)) \equiv u^2(\bar{f}_u\bar{\varphi}_r - \bar{f}_r\bar{\varphi}_u) \equiv 0.$$

Since $u \neq 0$, it follows that

$$\bar{f}_u\bar{\varphi}_r - \bar{f}_r\bar{\varphi}_u \equiv 0.$$

But

$$s = -(\bar{\varphi}_r/\bar{\varphi}_u).$$

Therefore, when $r = r''(u)$,

$$sf_u + f_r \equiv u(s\bar{f}_u + \bar{f}_r) \equiv u\left(\frac{-\bar{\varphi}_r\bar{f}_u}{\bar{\varphi}_u} + \bar{f}_r\right) \equiv (-u/\bar{\varphi}_u)(\bar{f}_u\bar{\varphi}_r - \bar{f}_r\bar{\varphi}_u) \equiv 0.$$

Therefore the transformation (12) is completely determined. Its Jacobian $1 - r_us + s_uv$ has the value 1 when $u = 0, v = 0$. Consequently (12) is one-to-one and analytic both ways. Since it further carries origin into origin, the results of this substitution in $f(u', v')$ and $\varphi(u', v')$ will each be analytic and take the value zero when $u = 0, v = 0$. The terms involving u alone are

$$f(u, r''(u)) \equiv c_{30}u^3 + Pu^4 + \dots, \quad \varphi(u, r''(u)) \equiv d_{20}u^2 + Qu^3 + \dots, \quad d_{20} \neq 0.$$

The coefficient of v^2 is computed to be

$$\begin{aligned} &\frac{s^2}{2}a_u + \left[\frac{s^2}{2}ra_{u^2}' + sa_u'\right] + \left[\frac{s^2r^2}{2}a_{u^2}'' + 2rsa_u'' + a''\right] \\ &\quad + \left[\frac{s^2}{2}r^3a_{u^2}''' + 3sr^2a_u''' + 3ra'''\right] + \dots \\ (21) \quad &= \frac{s^2}{2}[a_{u^2} + a_{u^2}'r + a_{u^2}''r^2 + a_{u^2}'''r^3 + \dots] \\ &\quad + s[a_u' + 2a_u''r + 3a_u'''r^2 + \dots] + [a'' + 3a'''r + \dots] \\ &= \frac{s^2}{2}f_{u^2}(u, r) + sf_{ur}(u, r) + \frac{1}{2}f_{r^2}(u, r), \end{aligned}$$

with a corresponding expression for the coefficient of v^2 in φ . When we replace r and s by their computed values from (19) and (20) and set $u = 0$, this coefficient for v^2 becomes

$$\text{for the transform of } f \dots \frac{1}{2} \cdot 2 = 1,$$

$$\text{for the transform of } \varphi \dots \frac{1}{2} \cdot 2d_{02} \neq 0.$$

We have then brought T' to the form

$$(22) \quad \begin{aligned} x &= (c_{30}u^3 + Pu^4 + \dots) + v^2(1 + e_{10}u + e_{01}v + \dots), \\ y &= (d_{20}u^2 + Qu^3 + \dots) + v^2(d_{02} + \dots). \end{aligned}$$

Set

$$(23) \quad u_1 = u\sqrt{d_{20} + Qu + \dots}, \quad v_1 = v\sqrt{d_{02} + \dots},$$

a transformation one-to-one and analytic both ways. Then (22) becomes

$$(24) \quad x = F(u_1) + v_1^2 G(u_1, v_1), \quad y = u_1^2 + v_1^2,$$

where $G(0, 0) \neq 0$, and $F(u_1)$ has u_1^3 as a factor. (For the second equation of (23) makes it evident that $v = 0$ when $v_1 = 0$, $u_1 = u$. Since further the Jacobian for the inverse of (23) can not vanish in the origin, it follows that this inverse must have the form

$$u = g(u_1), \quad v = v_1 h(u_1, v_1), \quad h(0, 0) \neq 0.$$

The conclusions concerning (24) are then immediate.) To find the inverse of (24) we have to solve for u and v the equations

$$(25) \quad F(u) + v^2 G(u, v) - x = 0, \quad u^2 + v^2 - y = 0, \quad G(0, 0) \neq 0,$$

where we have dropped the subscripts on u and v . The solutions v of the first equation, which are zero when $u = 0$, $x = 0$, using the Weierstrassian preparation theorem, are had from

$$(26) \quad v^2 + C(u, x)v + D(u, x) = 0,$$

where

$$(27) \quad F(u) + v^2 G(u, v) - x \equiv [v^2 + C(u, x)v + D(u, x)]e^{w(u, v, x)}.$$

The left number of (27) expressed as a power series in u, v, x contains no term involving v to the first power only. Consequently the identity (27) can hold true only if $C(u, x) \equiv 0$. Therefore solutions of (25) are had from

$$(28) \quad v^2 + D(u, x) = 0, \quad v^2 + u^2 - y = 0.$$

Eliminating v , we have

$$E(u, x, y) \equiv u^2 - D(u, x) - y = 0.$$

From (27) we find that

$$D(0, 0) = 0, \quad D_u(0, 0) = 0, \quad D_{u^2}(0, 0) = 0,$$

whence it follows at once that

$$E(0, 0, 0) = 0, \quad E_u(0, 0, 0) = 0, \quad E_{u^2}(0, 0, 0) = 2, \quad E(u, 0, 0) \neq 0.$$

Since

$$E_y(0, 0, 0) = -1,$$

$E(u, x, y)$ is irreducible. Consequently the discriminant $\Delta(x, y)$ of $E(u, x, y)$, which can be obtained by eliminating u from $E(u, x, y)$ and $E_u(u, x, y)$, is not identically zero. But $\Delta(0, 0) = 0$, therefore $\Delta(x, y) = 0$ defines a locus passing through the origin in the xy -space. $E(u, x, y) = 0$ defines u as a two-valued function of x and y , continuous throughout a complete neighborhood of $x = 0, y = 0$, analytic with two distinct determinations except for points of $\Delta(x, y) = 0$, where the two determinations of the root become coincident. Since

$$E(u(x, y), x, y) \equiv 0,$$

i. e., since

$$[u(x, y)]^2 - y \equiv D(u(x, y), x),$$

the form of (28) shows that the determination of v from either equation of (28) will satisfy the other equation also. v , the square root of a function continuous throughout the complete neighborhood of the origin, is itself continuous throughout the same region. It is four-valued in any point in which $[u(x, y)]^2 - y$ is two-valued, two-valued in any point in which $[u(x, y)]^2 - y$ is single-valued and different from zero, and has the value zero when

$$[u(x, y)]^2 - y = 0.$$

$x = 0, y = 0$ is a point of this last locus. In the neighborhood of any point of this region not on

$$\Delta(x, y) = 0,$$

the equations (28) have four distinct analytic solutions, which we may indicate schematically as

$$(u_0, v_0), (u_0, v_1), (u_1, v_2), (u_1, v_3)$$

or

$$(u_0, v_0), (u_0, v_1), (u_1, v_0), (u_1, v_1).$$

Hence the theorem is proved.

If instead of the transformation T we consider the transformation T_0 whose definition differs from that for T in that it omits the condition (c), we can deduce a theorem of which Theorem 6 is a special case.

Theorem 7. *If the transformation T_0 has the form*

$$x = f(u, v) \equiv f_m(u, v) + f_{m+1}(u, v) + \cdots,$$

$$y = \varphi(u, v) \equiv \varphi_n(u, v) + \varphi_{n+1}(u, v) + \cdots,$$

where $f_i(u, v)$ and $\varphi_i(u, v)$ are homogeneous polynomials of degree i in u and v ($i = m, m + 1, \cdots; i = n, n + 1, \cdots$), and where $f_m(u, v)$ and $\varphi_n(u, v)$ have no common factor, then there exists an mn -valued inverse, defined through-

out the complete neighborhood of $x = 0, y = 0$. This inverse is analytic with mn distinct branches except (if $mn > 1$) along a complex one-dimensional locus where it is continuous and less than mn -valued. Finally $u = 0, v = 0$ when $x = 0, y = 0$.

Since $f_m(u, v)$ and $\varphi_n(u, v)$ have no common factor, it follows a fortiori that $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$. But I have shown in the theorem quoted near the end of section 4 that if $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$ and if R be suitably restricted, then there exists an inverse defined throughout the complete neighborhood of $x = 0, y = 0$, everywhere continuous in that neighborhood, finitely multiple-valued, analytic except (if more than one-valued) along a complex one-dimensional locus where roots elsewhere distinct become coincident, and having the value $u = 0, v = 0$ when $x = 0, y = 0$. The proof of this is independent of any assumption concerning the value of $J(u, v)$ in R .*

$J(u, v)$ can not be identically zero; for if it were then y would be expressible† as an analytic function of x alone in a suitable neighborhood of $x = x_1 = f(u_1, v_1)$, where (u_1, v_1) is a point of R in which a first derivative of $f(u, v)$ or of $\varphi(u, v)$, say $f_u(u, v)$, is not zero. The same point (x, y) would then correspond to every point (u, v) which is a point of the locus $f(u, v) = x_1$ and is within a suitable neighborhood S of (u_1, v_1) in which $f_u(u, v) \neq 0$. This contradicts the existence of a finitely multiple-valued inverse, hence $J(u, v) \neq 0$.

If $J(0, 0) \neq 0$, then $m = n = 1$ and this inverse is single-valued and analytic throughout the region $\bar{R}$ in which it is defined; if $J(0, 0) = 0$, it is multiple valued and satisfies the properties required by the theorem except that as yet we have no count of the number of roots included in this inverse. For the case under discussion, in which $f_m(u, v)$ and $\varphi_n(u, v)$ have no common factor, this number is known‡ to be exactly mn . Further, in any point of the region of definition in which $J(u, v) \neq 0$, these roots are distinct. Hence the theorem is proved.

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*Transactions, loc. cit., p. 333.

†Jordan, Cours d'Analyse, vol. 1, 2d ed. (1893), p. 86.

‡G. A. Bliss, Transactions of the American Mathematical Society, Vol. XIII (1912), p. 134.